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# AN ASYMPTOTIC METHOD OF SOLVING PROBLEMS OF WAVE PROPAGATION IN STRINGS†

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An asymptotic method of solving problems of the propagation of waves in strings is proposed which uses the value of the characteristic deformation as the small parameter. The method is demonstrated using the example of the point action on a fixed particle of the string, which models it by a non-stationary displacement (this action is usually called a transverse shock [1, 2].

In the case considered, the transverse waves that propagate along the string with velocity $a_0$ produce a corresponding tension in front of the transverse head wave, the velocity of which is much less than $a_0$. The equation for the momentum in the initial direction of the string in the zeroth approximation leads to the fact that the tension is independent of the coordinate in the region where the transverse waves propagate. After determining the transverse displacements in this region the field of the longitudinal velocities and the longitudinal deformations are found, where the longitudinal and transverse components of the deformation in the zeroth approximation considerably exceed the overall deformation in value (their order of magnitude is lower than the order of magnitude of the latter).

For the case of a power dependence of the velocity of the applied force (a transverse shock) on time, when there is no initial tension in the string, the field of the transverse displacements has a self-similar form.

1. SUPPOSE that at the instant of time $t=0$ a point action (an impulse) is applied to a flexible string of infinite length situated along the $x$ axis, leading to displacement of the point $s_0=0$ as given by the relation $x=x_0(t)$, $y=y_0(t)$ ($s_0$ is the Lagrangian coordinate). The equations of motion of the string have the form [1]

$$\rho_0\frac{\partial^2 x}{\partial t^2}=\frac{\partial}{\partial s_0}(T\cos\varphi),\quad \rho_0\frac{\partial^2 y}{\partial t^2}=\frac{\partial}{\partial s_0}(T\sin\varphi) \tag{1.1}$$

Here $\rho_0$ is the initial density of the string, assumed constant, $x$, $y$ are the displacements of the point $s_0$, $T$ is the tension, which is a function of the strain $e$, $\varphi$ is the angle of inclination of the element of the string to its initial direction, and

$$e=\left[\left(1+\frac{\partial x}{\partial s_0}\right)^2+\left(\frac{\partial y}{\partial s_0}\right)^2\right]^{1/2}-1,\quad \cos\varphi=\left(1+\frac{\partial x}{\partial s_0}\right)(1+e)^{-1},$$

$$\sin\varphi=\frac{\partial y}{\partial s_0}(1+e)^{-1}$$

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The characteristics of system (1.1) and the relations for them have the following form [1]

$$ds_0/dt = \pm a, \quad \cos\varphi du + \sin\varphi dv = \pm a(\cos\varphi d\mu + \sin\varphi d\vartheta)$$
$$ds_0/dt = \pm \lambda, \quad \cos\varphi dv - \sin\varphi du = \pm \lambda(\cos\varphi d\vartheta - \sin\varphi d\mu) \tag{1.2}$$
$$a^2 = \frac{1}{\rho_0}\frac{dT}{de}, \quad \lambda^2 = \frac{T}{\rho_0(1+e)}, \quad u = \frac{\partial x}{\partial t}, \quad v = \frac{\partial y}{\partial t}, \quad \mu = \frac{\partial x}{\partial s_0}, \quad \vartheta = \frac{\partial y}{\partial s_0}$$

If we assume that when $t=0$ we have $u=u_0=\text{const}$, $\mu=\mu_0$, $v=\vartheta=0$, then since the line $t=0$ is not a characteristic, $|\lambda|<|a|$, the solution in the region $0<t<s_0/a_0$ will be $u=u_0$, $v=\vartheta=0$, $\mu=\mu_0$ (see Fig. 1). Here $s_0>0$, $a_0=a(\mu_0)$. This result can be obtained in two ways: either by taking into account the fact that the solution obtained satisfies system (1.2), the initial conditions, and is unique (for the Cauchy problem), or by writing (1.2) in finite differences and solving (taking into account the homogeneity of the initial conditions when $\varphi=0$).

Behind the elastic wave $s_0=a_0t$ it follows from the relations on the characteristics $ds_0/dt=\pm\lambda$ written in finite differences, taking into account the fact that $\varphi(A)=\varphi(B)=0$, that

$$v(P)=\lambda_0\vartheta(P), \quad v(P)=\vartheta(P)=0, \quad \varphi(P)=0 \tag{1.3}$$

if the point $P$ is sufficiently close to the points $A$ and $B$.

From the relation on the characteristic $ds_0/dt=-a_0$ we obtain

$$u(P)=-a_0[\mu(P)-\mu_0]+u_0 \tag{1.4}$$

Relations (1.3) and (1.4) will obviously be satisfied for all points on the characteristic $s_0-s_0(P)=a_0[t-t(P)]$. Taking the point $Q$ on this characteristic and drawing the characteristics $ds_0/dt=\lambda(P)$, $ds_0/dt=-\lambda(Q)$ through points $P$ and $Q$ when the deformations are equal $\mu(P)=\mu(Q)$, and consequently $\lambda(P)=\lambda(Q)$, we again obtain $v(R)=\vartheta(R)=0$; $u(R)=-a_0[\mu(R)-\mu_0]+u_0$. Hence, it can be shown that in the region bounded by their characteristics $s_0=a_0t$ and the characteristic $s_0=s_0^*(t)$, $s_0^*(0)=0$, where $ds_0^*/dt=\lambda$, the string has no transverse velocities and deformations, while the longitudinal velocities and deformations are connected by the relation

$$u=-a_0(\mu-\mu_0)+u_0 \tag{1.5}$$

**2.** The extent of the region of propagation of transverse waves is much narrower than for longitudinal waves, since $\lambda\sim a_0\sqrt{(e_0)}$, $s_0^*\sim a_0t\sqrt{(e_0)}$. In view of the fact that $e_0\ll 1$, $\cos\varphi\sim 1$, $\varphi\sim\vartheta$, from the first equation (1.1), taking into account the fact that $u\sim a_0e_0$, we obtain

$$T(s_0,t)-T(s_0^*,t)=\int_{s_0^*}^{s_0}\rho_0\frac{\partial u}{\partial t}ds_0, \quad e(s_0,t)-e(s_0^*,t)\sim e_0^{3/2}$$

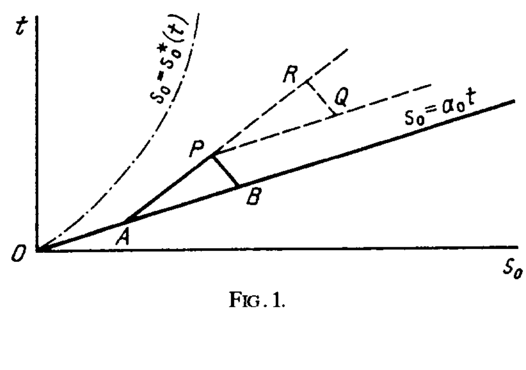


FIG. 1.

Consequently, in the principal approximation $e(s_0, t)=e(s_0^*, t)$, which reduces the second equation of (1.1) to an equation of small transverse oscillations, the velocity of propagation of which depends on time. This result can be obtained by the method of matched asymptotic expansions, if we change to new functions and independent variables $\bar{x}=x/(a_0t_0, e_0)$, $\bar{y}=y/(V_0t_0)$; $T=Ee_0\bar{e}$, $z=s_0/\bar{s}_0$, $\bar{t}=t/t_0$, $\bar{s}_0=e_0a_0t_0$ and seek the limiting form of Eqs (1.2) as $e_0\to 0$ assuming the functions introduced and their derivatives to be finite. The first equation of (1.1) then takes the form

$$e_0^{1/2}\frac{\partial^2\bar{x}}{\partial\bar{t}^2}=\frac{\partial\bar{e}}{\partial z} \tag{2.1}$$

and as $e_0\to 0$ for the principal approximation gives $\partial\bar{e}/\partial z=0$, $\bar{e}=\bar{e}(\bar{t})$. We will write the second equation as

$$\frac{\partial^2\bar{y}}{\partial\bar{t}^2}=\bar{e}(\bar{t})\frac{\partial^2\bar{y}}{\partial z^2} \tag{2.2}$$

The expression for the deformation

$$e_0\bar{e}\approx e_0^{1/2}\frac{\partial\bar{x}}{\partial z}+\frac{V_0^2}{2a_0^2e_0}\left(\frac{\partial\bar{y}}{\partial z}\right)^2 \tag{2.3}$$

shows that the first term in the principal approximation has an order that is much lower than $e_0$, and should be of the same order as the second term, and, moreover, equal to the principal approximation of the latter with opposite sign. Otherwise either $\partial\bar{y}/\partial z=0$ or $\partial\bar{x}/\partial z=0$, which gives no solution of the problem in question. Consequently, $\bar{V}_0=V_0\cdot a_0^{-1}\sim e_0^{3/4}$. Note that in [2], where the problem of a shock along a string with constant velocity was formulated for the first time, a similar dependence of the velocity on the deformation was found after the solution was obtained.

Thus, from the relation

$$\frac{\partial\bar{x}}{\partial z}=-\frac{V_0^2}{2a_0^2e_0^{3/2}}\left(\frac{\partial\bar{y}}{\partial z}\right)^2 \tag{2.4}$$

after solving Eq. (2.2) on the assumption that $\bar{y}_0(\bar{t})$, $\bar{e}(\bar{t})$ and $\bar{x}_0(\bar{t})$ are specified, we obtain the principal approximation for $\bar{x}(z, t)$. From (1.5) we can determine the relation between the displacement $\bar{y}_0(\bar{t})$ and the deformation $\bar{e}(\bar{t})$.

From Eq. (2.1) we can further determine the variation of the overall deformation in the region of transverse motion, from (2.2) we obtain the next term of the asymptotic expansion for $\bar{y}$, and then, from (2.3), we can obtain the next term of the asymptotic expansion for $\bar{x}$, etc.

**3.** We will construct a solution of the problem for the case when the initial deformation of the string $\mu_0=0$ and $y_0(t)=At^m$ $(m\geqslant 1)$ (the velocity of the shock $y_0^{\cdot}=Amt^{m-1}$). We will use the semi-inverse method, assuming that the variation of the overall deformation in this case is given by the power relation $e=Bt^n$.

For the case considered

$$e\ll 1,\quad ds_0^*/dt=a_0B^{1/2}t^{n/2},\quad s_0^*=a_0B^{1/2}(1+n/2)^{-1}t^{1+n/2}$$

Note that for a given $e(t)$ the conditions $y(0, t)=y_0(t)$, $y(s_0^*, t)=0$ define a unique solutions of Eq. (2.2), which when $e(t)=Bt^n$, $y_0=At^m$ will be the function $y=At^mf(z)$, $z=s_0/s_0^*$.

The function $f(z)$ satisfies the equation and boundary conditions

$$m(m-1)f-(1+n/2)(2m-2-n/2)zf'=(1+n/2)^2(1-z^2)f'' \tag{3.1}$$
$$f(0)=1,\quad f(1)=0$$

We will consider the behaviour of $f(z)$ as $z \to 1$, by writing

$$f(z) = A_1 (1 - z)^N + \ldots$$

Then $N = (2m + n/2)(2+n)^{-1}$ and when $N > 1$ we have $f'(1) = 0$.

When $N > 1$ there is no discontinuity on the transverse head wave, and hence the longitudinal components of the velocity on it are not disrupted.

From (2.4) we obtain

$$x = x_0(t) - \frac{A^2}{2a_0\sqrt{B}}(1 + n/2)\, t^{2m-1-n/2} \int_0^z f'^2 dz$$

On the head wave $z = 1$ when $N > 1$, in the case when $x_0' = 0$, which we will use to simplify the calculations, the longitudinal component of the velocity

$$\left(\frac{\partial x}{\partial t}\right)_{z=1} = -\frac{A^2(1 + n/2)(2m - 1 - n/2)}{2a_0\sqrt{B}}\, t^{2m-2-n/2} \int_0^1 f'^2 dz = -a_0 e(t)$$

Hence $n = 4/3(m-1)$, i.e. $e \sim y_0'^{4/3}$, which might have been expected in accordance with the asymptotic problem. In this case, $N - 1 = 2(m-1)(2m-1)^{-1} > 0$ when $m > 1$, which indicates that there is no break on the transverse head wave for all $m > 1$, but there is when $m = 1$. When $m > 1$ the condition of continuity of the longitudinal velocities when $z = 1$ leads to the following relationship between the coefficients in the velocity and deformation relations

$$(1 + 2m)(4m - 1)A^2 \int_0^1 f'^2 dz = 18a_0^2 B^{3/2}$$

Equation (3.1) then takes the form

$$Lf = m(m - 1)f - {}^4/_9(1 + 2m)(m - 1)zf' - {}^1/_9(1 + 2m)^2(1 - z^2)f'' = 0 \qquad (3.2)$$

and can be solved numerically or approximately. In the latter case, we can use the method of integral relations, representing the solution approximately in the form

$$f(z) = c(1 - z)^N + (1 - c)(1 - z)^{N+1}$$

which ensures that the boundary conditions are satisfied when $z = 0$ and $z = 1$ and the asymptotic form when $z \to 1$.

By satisfying Eq. (3.2) integrally in the form

$$\int_0^1 Lf dz = 0$$

we obtain the unknown constant

$$C = 6m(23m^2 + 11m + 2)(2m + 1)^{-1}(73m^2 + m - 2)^{-1}$$

the method can be generalized to the problem of a shock on an elastic string by a blunt body and also to the problem of a shock on elastic membranes.

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